Fundamental Theorem, Conservative Vector Fields

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Suppose we have a curve/path C from $\vec{a}$ to $\vec{b}$.

$\vec{F}(x,y) = P(x,y)\,\vec{i}$

$\vec{r} = x\hat{i} + y\hat{j} = (x,y)$

$(d\vec{r} = (dx, dy)$ etc$)$

Then we defined $\displaystyle\int_C P\,dx + Q\,dy = \int_C \vec{F}\cdot d\vec{r}$

$$= \lim_{mesh \to 0} \sum_{i=1}^{N} P(x_i, y_i)\,\Delta x_i + Q(x_i, y_i)\,\Delta y_i$$

$$= \lim \sum \vec{F}(x_i, y_i)\cdot \Delta\vec{r}_i$$

where $\Delta x_i = x_i - x_{i-1}$ $\quad \Delta y_i = y_i - y_{i-1}$

mesh $= \max \| (x_i, y_i) - (x_{i-1}, y_{i-1}) \|$

(recall $\|\vec{v} - \vec{w}\|$ = distance from $\vec{v}$ to $\vec{w}$)

where

$\vec{a} = (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) \ldots, (x_{N-1}, y_{N-1}), (x_N, y_N) = \vec{b}$

is a sequence of points on C.

"$\lim_{mesh \to 0}$" means the limit over all such sequences as the mesh approaches 0.

i.e. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. the Riemann sum is w/in ε of the integral for any such sequence with mesh $< \delta$

Warning

① If we go from $\vec{b}$ to $\vec{a}$ along C (in the opposite direction), we call this path $-C$.

then $\displaystyle\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}$

Why? if $(x_0, y_0), (x_1, y_1), \ldots, (x_{N-1}, y_{N-1}), (x_N, y_N)$ is along C

then

$(x_N, y_N), (y_{N-1}, \ldots), \ldots, (x_1, y_1), (x_0, y_0)$ goes along $-C$

so

you negate the Δx_i.

② If C is a loop (closed loop) like $\vec{a} = \vec{b}$, then you must specify the direction of C. And if you reverse the direction, you get negative of the circle.

eg if C is a circle

Compare w/ single variable

$$\int_0^1 x^2\,dx = \frac{1}{3} \qquad \int_1^0 x^2\,dx = -\frac{1}{3}$$

$\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$ is true even if $a > b$.

How to compute?

Choose a parameterization, ie a pair of fcns $x(t)$, $y(t)$ defined for $t \in [a,b]$

s.t. $\vec{a} = (x(a), y(a))$

$\vec{b} = (x(b), y(b))$

and $(x(t), y(t))$ goes along the path C as t goes from a to b.

Now $\quad dx = \dfrac{dx}{dt}\,dt \qquad dy = \dfrac{dy}{dt}\,dt$

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \int P\frac{dx}{dt}\,dt + Q\frac{dy}{dt}\,dt$$

$$= \int_C \left(P\,x'(t) + Q\,y'(t)\right) dt$$

Note $\quad \displaystyle\int_C = \int_{t=a}^{t=b}$

eg $\quad P(x,y) = x^2 - y^2$

$\quad\;\; Q(x,y) = 3x - e^y$

Consider a segment C of a parabola:

eg $P(x,y) = x^2 - y^2$
$Q(x,y) = 3x - e^y$

Consider a segment C of a parabola:
given by
$(x,y) = (t, t^2)$ from $t=1$ to $t=2$

then $\int_C P\,dx + Q\,dy =$

$$= \int_{t=1}^{t=2} \left(P(x(t), y(t))\, x'(t) + (Q(x(t), y(t))\, y'(t) \right)$$

$$= \int_1^2 \left((t^2 - t^4) + (3t - e^{t^2})(2t) \right) dt$$

$$= \int_1^2 t^2 - t^4 + 6t^2 - 2te^{t^2}\, dt$$

$$= \left[\frac{7t^3}{3} - \frac{t^5}{5} - e^{t^2} \right]_{t=1}^{t=2}$$

$$= \left(\frac{56}{3} - \frac{32}{5} - e^4 \right) - \left(\frac{7}{3} - \frac{1}{5} - e \right)$$

$$= \frac{49}{3} - \frac{31}{5} + e - e^4$$

Note og definition did not depend on a parameterization.
eg what if we used
$(x,y) = (\sqrt{t}, t)$ for $t \in [1,4]$
$\Rightarrow$ the fact that there's a definition independent of parameterization implies that we get the same result

What about 3 dim?

Exact same formula:

$$\int_C \vec{F} \cdot d\vec{r} = \lim_{\text{mesh} \to 0} \sum_{i=1}^{N} \vec{F}(x_i, y_i, z_i) \cdot \Delta \vec{r}_i$$

now C is a path in $\mathbb{R}^3$
from $\vec{a} = (x_0, y_0, z_0)$
to $\vec{b} = (x_N, y_N, z_N)$
and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$
$\Delta \vec{r}_i = (\Delta x_i, \Delta y_i, \Delta z_i)$
and mesh $= \max_i \| (x_i, y_i, z_i) - (x_{i-1}, y_{i-1}, z_{i-1}) \|$

Note $\vec{F}$ must have 3 components bc we take its dot product with $\Delta \vec{r}_i$ and now $\Delta \vec{r}_i$ has 3 components

so $$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

In n-dim: $\vec{F}$ should have n outputs but C is still a path (1-dim) in $\mathbb{R}^n$

Note if $(x(t), y(t))$ for $t \in [a,b]$ is a parameterization of C
then $x(a+b-t), y(a+b-t)$ for $t \in [a,b]$ is a parametriz. of $-C$
Notice:
$(x(a+b-a), y(a+b-a)) = (x(b), y(b))$
$(x(a+b-b), y(a+b-b)) = (x(a), y(a))$
Q/ How does this negate the integral?
A/ bc it negates $x'(t)$ and $y'(t)$
ie. $\frac{d(x(a+b-t))}{dt} = -\frac{dx}{dt}$

4.1 [Co] $\rightarrow$ diff kind of line integration
instead of $\int_C \vec{F} \cdot d\vec{r}$ we do

$\int_C f\,ds$ where s is arclength

$$ds = \|dr\|$$
$$= \sqrt{dx^2 + dy^2}$$

In Riemann sum terms:

$$\int f\,ds = \lim_{m \to 0} \sum_{i=1}^{N} f(x_i, y_i)\,\Delta s_i$$

where $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \|(x_i, y_i) - (x_{i-1}, y_{i-1})\|$

How to calc?

$dx = \frac{dx}{dt}dt \quad dy = \frac{dy}{dt}dt$

so $ds = \sqrt{dx^2 + dy^2}$

How to calculate?

$dx = \frac{dx}{dt}dt \quad dy = \frac{dy}{dt}dt$

so $ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2}$

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$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\Longrightarrow \int_C f\,ds = \int f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Note

So $ds = \sqrt{dx^2+dy^2} = \sqrt{\left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)dt^2}$

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Note

$$\frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{T} \qquad \text{(Chap 1)}$$

$$\Rightarrow d\vec{r} = \underbrace{ds}_{\text{scalar}}\ \underbrace{\vec{T}}_{\text{vector}}$$

$$\Rightarrow \vec{f}\cdot d\vec{r} = \vec{f}\cdot\left(ds\,\vec{T}\right) = ds\left(\vec{f}\cdot\vec{T}\right)$$

$$\Rightarrow \int_C \vec{f}\cdot d\vec{r} = \int_C \left(\vec{f}\cdot\vec{T}\right) ds$$

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$$\Rightarrow \int_C \vec{f}\cdot d\vec{r} = \int_C \left(\vec{f}\cdot\vec{T}\right) ds = \int_C f\,ds$$

where $f = \vec{f}\cdot\vec{T}$ (dot prod of vectors is a scalar)

Recall $\vec{F}\cdot\vec{T}$ is the "component" of $\vec{F}$ in the $\vec{T}$-direction

so, eg if $\vec{F}$ and $\vec{T}$ are in the same direction, then $\vec{F}\cdot\vec{T} = \|\vec{F}\|$

if $\vec{F}$ and $\vec{T}$ are $\perp$, the $\vec{F}\cdot\vec{T} = 0$

Recall $\vec{T}$ is tangent to the path C.

In physics: work = force × distance (simple)

More sophisticated:

Force is a vector

displacement is a vector and

W = (force) • (displacement)

Force is a vector
displacement is a vector and
$W = (\text{force}) \cdot (\text{displacement})$

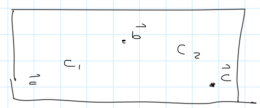
eg. If Force $\perp$ to the direction of motion (eg for an obj in circular orbit) then no work is done.

Line integrals $\int_C \vec{F} \cdot d\vec{r}$ allow us to calculate if $\vec{F}$ is the Force vector

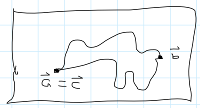
ie $\int_C \vec{F} \cdot d\vec{r}$ is the work done by the force done on an object as it goes along the path C. (usually t = time)

If C_1 is a path from $\vec{a}$ to $\vec{b}$ and C_2 a path from $\vec{b}$ to $\vec{c}$, the $C_1 + C_2$ is the path from $\vec{a}$ to $\vec{c}$ given by going along C_1 then C_2.

eg



eg suppose $\vec{a} = \vec{c}$



then $C_1 + C_2$ is from $\vec{a}$ to $\vec{c}$
↳ closed path (loop)

Next time

$$\int_{C_1 + C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$